DEFORMATIONS OF ASYMPTOTICALLY CYLINDRICAL COASSOCIATIVE SUBMANIFOLDS WITH MOVING BOUNDARY

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ABSTRACT. In an earlier paper, [5], we proved that given an asymptotically cylindrical G_2-manifold M with a Calabi–Yau boundary X, the moduli space of coassociative deformations of an asymptotically cylindrical coassociative 4-fold $C \subset M$ with a fixed special Lagrangian boundary $L \subset X$ is a smooth manifold of dimension $\dim V_+$, where V_+ is the positive subspace of the image of $H^2_{cs}(C, \mathbb{R})$ in $H^2(C, \mathbb{R})$. In order to prove this we used the powerful tools of Fredholm Theory for noncompact manifolds which was developed by Lockhart and McOwen [11], [12], and independently by Melrose [13], [14].

In this paper, we extend our result to the moving boundary case. Let $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{cs}(C, \mathbb{R})$ be the natural projection, so that $\ker \Upsilon$ is a vector subspace of $H^2(L, \mathbb{R})$. Let F be a small open neighbourhood of 0 in $\ker \Upsilon$ and L_s denote the special Lagrangian submanifolds of X near L for some $s \in F$ and with phase i. Here we prove that the moduli space of coassociative deformations of an asymptotically cylindrical coassociative submanifold C asymptotic to $L_s \times (R, \infty)$, $s \in F$, is a smooth manifold of dimension equal to $\dim V_+ + \dim \ker \Upsilon = \dim V_+ + b^2(L) - b^0(L) + b^3(C) - b^1(C) + b^0(C)$.

1. INTRODUCTION

Let (M, φ, g_M) be a connected, complete, asymptotically cylindrical G_2-manifold with a G_2-structure (φ, g_M) and asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$, where X is a Calabi–Yau 3-fold. An asymptotically cylindrical G_2-manifold M is a noncompact Riemannian 7-manifold with zero Ricci curvature whose holonomy group $\mathrm{Hol}(g_M)$ is a subgroup of the exceptional Lie group G_2. It is equipped with a covariant constant 3-form φ and a 4-form $*\varphi$. There are two natural classes of noncompact calibrated submanifolds inside M corresponding to φ and $*\varphi$ which are called asymptotically cylindrical associative 3-folds and coassociative 4-folds, respectively.

The Floer homology programs for asymptotically cylindrical Calabi-Yau and G_2-manifolds lead to the construction of brand new Topological Quantum Field Theories. In order to construct consistent TQFT's, the first step is to understand the deformations of asymptotically cylindrical calibrated submanifolds with some small decay rate inside Calabi-Yau and G_2-manifolds. A fundamental question is whether these noncompact submanifolds have smooth deformation spaces.

For this purpose, in [5] we studied the deformation space of an asymptotically cylindrical coassociative submanifold C in an asymptotically cylindrical G_2-manifold M with a Calabi-Yau boundary X. We assumed that the boundary $\partial C = L$ is a special Lagrangian submanifold of X. Using the analytic set-up which was developed for elliptic operators on

asymptotically cylindrical manifolds by Lockhart-McOwen and Melrose, [11], [12], [13], [14], we proved that for fixed boundary $\partial C = L$ this moduli space is a smooth manifold.

In order to develope the Floer homology program for coassociative submanifolds the next step is to parametrize coassociative deformations with moving boundary. In this paper, we extend our previous result in [5] to the moving boundary case and show that if the special Lagrangian boundary is allowed to move, one still obtains a smooth moduli space. Moreover, we can determine the dimension of this space. These two technical results on fixed and moving boundary cases then parametrize all asymptotically cylindrical coassociative deformations. Remarkably, using this parametrization of deformations and basic algebraic topology we can verify one of the main claims of Leung in [10] to prove that the boundary map from the moduli space of coasssociative cycles to the moduli space of special Lagrangian cycles is a Lagrangian immersion. With this result in hand, one can start defining the Floer homology of coassociative cycles with special Lagrangian boundary, which then leads to the construction of the TQFT of these cycles.

In this paper we prove the following theorem.

Theorem 1.1. *Let (M, φ, g) be an asymptotically cylindrical G_2-manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$, where X is a Calabi–Yau 3-fold with metric g_X. Let C be an asymptotically cylindrical coassociative 4-fold in M asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate β for $\alpha \leqslant \beta < 0$, where L is a special Lagrangian 3-fold in X with phase i, and metric $g_L = g_X|_L$.*

Let also $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{cs}(C, \mathbb{R})$ be the natural projection, so that $\ker \Upsilon$ is a vector subspace of $H^2(L, \mathbb{R})$ and let F be a small open neighbourhood of 0 in $\ker \Upsilon$. Let also L_s be the special Lagrangian submanifolds of X near L for some $s \in F$ and with phase i. Then for any sufficiently small γ the moduli space $\mathcal{M}^\gamma_C$ of asymptotically cylindrical coassociative submanifolds in M close to C, and asymptotic to $L_s \times (R', \infty)$ with decay rate γ, is a smooth manifold of dimension $\dim V_+ + \dim \ker \Upsilon = \dim V_+ + b_2(L) - b_0(L) + b_3(C) - b_1(C) + b_0(C)$, where V_+ is the positive subspace of the image of $H^2_{cs}(C, \mathbb{R})$ in $H^2(C, \mathbb{R})$.

Remark 1.2. In Theorem 1.1 above, as in the fixed boundary case, [5], we require γ to satisfy $\beta < \gamma < 0$, and $(0, \gamma^2]$ to contain no eigenvalues of the Laplacian Δ_L on functions on L, and $[\gamma, 0)$ to contain no eigenvalues of the operator $- * \mathrm{d}$ on coexact 1-forms on L. These hold provided $\gamma < 0$ is small enough. This assumption is needed to guarantee that the linearized operator of the deformation map for asymptotically cylindrical coassociative submanifolds is Fredholm.

Remark 1.3. In our previous paper, [5], we proved that the dimension of the deformation space of asymptotically cylindrical coassociative submanifold C with fixed special Lagrangian boundary L is given as V_+ where V_+ is the positive subspace of the image of $H^2_{cs}(C, \mathbb{R})$ in $H^2(C, \mathbb{R})$. One should note that in Theorem 1.1 above, in the case when L is a special Lagrangian homology 3-sphere, $b_1(L) = 0$ and hence L is rigid and there are no special Lagrangian deformations of L so it behaves like a fixed boundary. Then one can use basic algebraic topology to show that for this special case Theorem 1.1 gives us $\dim V_+ + \dim \ker \Upsilon = \dim V_+ + b_2(L) - b_0(L) + b_3(C) - b_1(C) + b_0(C) = \dim V_+$ which is consistent with the result of our previous paper, [5], for fixed boundary.

Remark 1.4. Theorem 1.1 applies to examples of asymptotically cylindrical G_2-manifolds constructed by Kovalev, in [9]. These 7-manifolds are of the form $X \times S^1$ and are obtained by the direct product of asymptotically cylindrical 6-manifolds X with holonomy $SU(3) \subset G_2$ and the circle S^1. It is still an open question whether there exist asymptotically cylindrical 7-manifolds with holonomy group G_2.

The outline of the paper is as follows: In §2, we begin with a general discussion of G_2 geometry, asymptotically cylindrical manifolds and basic tools used in elliptic theory such as weighted Sobolev spaces. In §3, we give an overview of deformations of an asymptotically cylindrical coassociative submanifold with fixed special Lagrangian boundary, followed by a sketch proof of our main theorem in [5], where we showed that the moduli space of such deformations is smooth and calculated its dimension. In §4 we introduce the analytic set-up for deformations with moving (free) boundary and prove Theorem 1.1. Finally, in §5 using Theorem 1.1, we verify one of the main claims of Leung in [10], which is necessary to prove that the boundary map from the moduli space of coassociative cycles into the moduli space of special Lagrangian cycles is a Lagrangian immersion.

2. $\mathbf{G_2}$-MANIFOLDS AND COASSOCIATIVE SUBMANIFOLDS

We now explain some background material on asymptotically cylindrical G_2-manifolds and their coassociative submanifolds. We will also review the Fredholm Theory for elliptic operators and weighted Sobolev spaces. A good reference for G_2 geometry is Harvey and Lawson [7]. The details of the elliptic theory on noncompact manifolds can be found in [11],[12],[13],[14].

2.1. **$\mathbf{G_2}$ Geometry.** The imaginary octonions $Im\mathbb{O} = \mathbb{R}^7$ is equipped with the cross product $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by $u \times v = Im(u \cdot \bar{v})$ where $\cdot$ is the octonionic multiplication. The exceptional Lie group G_2 can be defined as the linear automorphisms of $Im\mathbb{O}$ preserving this cross product operation. It is a compact, semisimple, and 14-dimensional subgroup of $SO(7)$.

Definition 2.1. A smooth 7-manifold M has a G_2-structure if its tangent frame bundle reduces to a G_2 bundle.

It is known that if M has a G_2-structure then there is a G_2-invariant 3-form $\varphi \in \Omega^3(M)$ which can be written in an orthonormal frame as

$$\varphi = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^4 \wedge e^5 + e^1 \wedge e^6 \wedge e^7 + e^2 \wedge e^4 \wedge e^6 - e^2 \wedge e^5 \wedge e^7$$
$$-e^3 \wedge e^4 \wedge e^7 - e^3 \wedge e^5 \wedge e^6.$$

This G_2-invariant 3-form φ gives an orientation $\mu \in \Omega^7(M)$ on M and μ determines a metric $g = g_\varphi = \langle \, , \, \rangle$ on M defined as

$$\langle u, v \rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi]/\mu,$$

where $i_v = v \lrcorner$ is the interior product with a vector v. So from now on we will refer to the term (φ, g) as a G_2-structure and the term (M, φ, g) as a manifold with G_2-structure.

Definition 2.2. (M, φ, g) is a G_2-manifold if $\nabla \varphi = 0$, i.e. the G_2-structure φ is torsion-free.

One can show that for a manifold with G_2-structure (M, φ, g), the following are equivalent:

   (i) $\mathrm{Hol}(g) \subseteq G_2$,
   (ii) $\nabla \varphi = 0$ on M, where ∇ is the Levi-Civita connection of g,
   (iii) $\mathrm{d}\varphi = 0$ and $\mathrm{d}^* \varphi = 0$ on M.

Harvey and Lawson, [7], showed that there are minimal submanifolds of G_2-manifolds calibrated by φ and $*\varphi$.

Definition 2.3. Let (M, φ, g) be a G_2-manifold. A 4-dimensional submanifold $C \subset M$ is called *coassociative* if $\varphi|_C = 0$. A 3-dimensional submanifold $Y \subset M$ is called *associative* if $\varphi|_Y \equiv vol(Y)$; this condition is equivalent to $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w, z)$$

2.2. Asymptotically Cylindrical G_2-Manifolds and Coassociative Submanifolds.
Next, we recall basic properties of asymptotically cylindrical G_2-manifolds and the coassociative 4-folds. We also need these definitions and the analytic set-up in Section 4. More details on the subject can be found in [5].

Definition 2.4. A G_2-manifold (M_0, φ_0, g_0) is called cylindrical if $M_0 = X \times \mathbb{R}$ and (φ_0, g_0) is compatible with the product structure, that is,

$$\varphi_0 = \mathrm{Re}\, \Omega + \omega \wedge \mathrm{d}t \qquad \text{and} \qquad g_0 = g_X + \mathrm{d}t^2,$$

where X is a connected, compact Calabi–Yau 3-fold with Kähler form ω, Riemannian metric g_X and holomorphic (3,0)-form Ω.

Definition 2.5. Let X be a Calabi–Yau 3-fold with metric g_X. (M, φ, g) is called an asymptotically cylindrical G_2-manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$ if there exists a cylindrical G_2-manifold (M_0, φ_0, g_0) with $M_0 = X \times \mathbb{R}$, a compact subset $K \subset M$, a real number R, and a diffeomorphism $\Psi : X \times (R, \infty) \to M \setminus K$ such that $\Psi^*(\varphi) = \varphi_0 + \mathrm{d}\xi$ for some smooth 2-form ξ on $X \times (R, \infty)$ with $\left| \nabla^k \xi \right| = O(e^{\alpha t})$ on $X \times (R, \infty)$ for all $k \geqslant 0$, where ∇ is the Levi-Civita connection of the cylindrical metric g_0.

An asymptotically cylindrical G_2-manifold M has one end modelled on $X \times (R, \infty)$, and as $t \to \infty$ in (R, ∞) the G_2-structure (φ, g) on M converges to order $O(e^{\alpha t})$ to the cylindrical G_2-structure on $X \times (R, \infty)$, with all of its derivatives. As in [5], we suppose M and X are connected, that is, we allow M to have only one end. We showed earlier that an asymptotically cylindrical G_2-manifold can have at most one cylindrical end, or otherwise the holonomy reduces, [16].

We can also define calibrated submanifolds of noncompact G_2-manifolds. In [5], we introduced definitions of cylindrical and asymptotically cylindrical coassociative submanifolds of a G_2-manifold.

Definition 2.6. Let (M_0, φ_0, g_0) be a cylindrical G_2-manifold. A 4-fold C_0 is called cylindrical coassociative submanifold of M_0 if $C_0 = L \times \mathbb{R}$ for some compact special Lagrangian 3-fold L with phase i in the Calabi-Yau manifold X.

Definition 2.7. Let M be an asymptotically cylindrical G_2-manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$. Let also C be a connected, complete asymptotically cylindrical 4-fold in M asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate β for $\alpha \leqslant \beta < 0$ and let L be a compact special Lagrangian 3-fold in X with phase i, and metric $g_L = g_X|_L$. Then C is called asymptotically cylindrical coassociative 4-fold if there exists a compact subset $K' \subset C$, a normal vector field v on $L \times (R', \infty)$ for some $R' > R$, and a diffeomorphism $\Phi : L \times (R', \infty) \to C \setminus K'$ such that the following diagram commutes:

(1)
$$
\begin{array}{ccccc}
X \times (R', \infty) & \xleftarrow{\exp_v} & L \times (R', \infty) & \xrightarrow{\Phi} & (C \setminus K') \\
\downarrow{\scriptstyle\subset} & & & & \downarrow{\scriptstyle\subset} \\
X \times (R, \infty) & & \xrightarrow{\hspace{2em}\Psi\hspace{2em}} & & (M \setminus K),
\end{array}
$$

and $\left|\nabla^k v\right| = O(e^{\beta t})$ on $L \times (R', \infty)$ for all $k \geqslant 0$.

Diagram (1) implies that C in M is asymptotic to the cylinder C_0 in $M_0 = X \times \mathbb{R}$ as $t \to \infty$ in $\mathbb{R}$, to order $O(e^{\beta t})$. In other words, C can be written near infinity as the graph of a normal vector field v to $C_0 = L \times \mathbb{R}$ in $M_0 = X \times \mathbb{R}$, so that v and its derivatives are $O(e^{\beta t})$. Here we require C but not L to be connected, so it is possible that C to have multiple ends.

2.3. **Elliptic operators on noncompact manifolds.** We will conclude this section with a brief review of the weighted Sobolev spaces and the results of Lockhart and McOwen, [11], [12], about Fredholm properties of elliptic operators on manifolds with cylindrical ends.

Let C and L be as in Definition 2.7 above. E_0 is called a cylindrical vector bundle on $L \times \mathbb{R}$ if it is invariant under translations in $\mathbb{R}$. Let h_0 be a smooth family of metrics on the fibres of E_0 and ∇_{E_0} a connection on E_0 preserving h_0, with h_0, ∇_{E_0} invariant under translations in $\mathbb{R}$. Let E be a vector bundle on C equipped with metrics h on the fibres, and a connection ∇_E on E preserving h. We say that E, h, ∇_E are asymptotic to E_0, h_0, ∇_{E_0} if there exists an identification $\Phi_*(E) \cong E_0$ on $L \times (R', \infty)$ such that $\Phi_*(h) = h_0 + O(e^{\beta t})$ and $\Phi_*(\nabla_E) = \nabla_{E_0} + O(e^{\beta t})$ as $t \to \infty$. Then we call E, h, ∇_E asymptotically cylindrical.

Definition 2.8. Let $\rho : C \to \mathbb{R}$ be a smooth function satisfying $\Phi^*(\rho) \equiv t$ on $L \times (R', \infty)$. For $p \geq 1$, $k \geq 0$ and $\gamma \in \mathbb{R}$ the weighted Sobolev space $L^p_{k,\gamma}(E)$ is defined to be the set of sections s of E that are locally integrable and k times weakly differentiable and for which the norm

(2)
$$
\|s\|_{L^p_{k,\gamma}} = \Big(\sum_{j=0}^{k} \int_C e^{-\gamma \rho} \big|\nabla_E^j s\big|^p \mathrm{d}V\Big)^{1/p}
$$

is finite.

Note that the weighted Sobolev space, $L_{k,\gamma}^p(E)$, is a Banach space.

Now suppose E, F are two asymptotically cylindrical vector bundles on C, asymptotic to cylindrical vector bundles E_0, F_0 on $L \times \mathbb{R}$.

Definition 2.9. Let $A_0 : C^\infty(E_0) \to C^\infty(F_0)$ be a cylindrical elliptic operator of order k invariant under translations in $\mathbb{R}$. Let also $A : C^\infty(E) \to C^\infty(F)$ be an elliptic operator of order k on C. A is asymptotic to A_0 if under the identifications $\Phi_*(E) \cong E_0$, $\Phi_*(F) \cong F_0$ on $L \times (R', \infty)$ we have $\Phi_*(A) = A_0 + O(e^{\beta t})$ as $t \to \infty$ for $\beta < 0$. Then A is called an asymptotically cylindrical elliptic operator.

It is well known that if A is an elliptic operator on a compact manifold then it should be Fredholm. But this is not the case for the noncompact manifolds; there are examples of elliptic operators which are not Fredholm, [11], [12] and it turns out that A is Fredholm if and only if γ does not lie in a discrete set $\mathcal{D}_{A_0} \subset \mathbb{R}$ which can be defined as follows:

Definition 2.10. Let A and A_0 be elliptic operators on C and $L \times \mathbb{R}$, so that E, F have the same fibre dimensions. Extend A_0 to the complexifications $A_0 : C^\infty(E_0 \otimes_\mathbb{R} \mathbb{C}) \to C^\infty(F_0 \otimes_\mathbb{R} \mathbb{C})$. Define $\mathcal{D}_{A_0}$ to be the set of $\gamma \in \mathbb{R}$ such that for some $\delta \in \mathbb{R}$ there exists a nonzero section $s \in C^\infty(E_0 \otimes_\mathbb{R} \mathbb{C})$ invariant under translations in $\mathbb{R}$ such that $A_0(e^{(\gamma+i\delta)t}s) = 0$.

An important Fredholm property of elliptic operators on noncompact manifolds has been shown by Lockhart and McOwen in [12, Th. 1.1]:

Theorem 2.11. *Let (C, g) be an asymptotically cylindrical Riemannian manifold asymptotic to $(L \times \mathbb{R}, g_0)$, and $A : C^\infty(E) \to C^\infty(F)$ an asymptotically cylindrical elliptic operator on C of order k between asymptotically cylindrical vector bundles E, F on C, asymptotic to the cylindrical elliptic operator $A_0 : C^\infty(E_0) \to C^\infty(F_0)$ on $L \times \mathbb{R}$. Let $\mathcal{D}_{A_0}$ be defined as above.*

Then $\mathcal{D}_{A_0}$ is a discrete subset of $\mathbb{R}$, and for $p > 1$, $l \geqslant 0$ and $\gamma \in \mathbb{R}$, the extension $A_{k+l,\gamma}^p : L_{k+l,\gamma}^p(E) \to L_{l,\gamma}^p(F)$ is Fredholm if and only if $\gamma \notin \mathcal{D}_{A_0}$.

Note that Theorem 2.11 plays an important role in our choice of weighted Sobolev spaces in Theorem 1.1.

3. Coassociative Deformations with Fixed Boundary

Using the set-up in Section 2, we proved the following theorem in [5], for asymptotically cylindrical coassociative deformations with fixed boundary.

Theorem 3.1. [5, Thm 1.1.] *Let (M, φ, g) be an asymptotically cylindrical G_2-manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$, where X is a Calabi–Yau 3-fold with metric g_X. Let C be an asymptotically cylindrical coassociative 4-fold in M asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate β for $\alpha \leqslant \beta < 0$, where L is a special Lagrangian 3-fold in X with phase i, and metric $g_L = g_X|_L$.*

Then for some small γ the moduli space $\mathcal{M}_C^\gamma$ of asymptotically cylindrical coassociative submanifolds in M close to C, and asymptotic to $L \times (R', \infty)$ with decay rate γ, is a smooth

manifold of dimension $\dim V_+$, *where* V_+ *is the positive subspace of the image of* $H^2_{\mathrm{cs}}(C, \mathbb{R})$ *in* $H^2(C, \mathbb{R})$.

Remark 3.2. McLean proved the compact version of Theorem 3.1 in [15] and showed that the moduli space $\mathcal{M}_C$ of coassociative 4-folds isotopic to a compact coassociative 4-fold C in M is a smooth manifold of dimension $b^2_+(C)$. There he modelled $\mathcal{M}_C$ on $\tilde{P}^{-1}(0)$ for a nonlinear map $\tilde{P}$ between Banach spaces, whose linearization $\mathrm{d}\tilde{P}(0, 0)$ at 0 was the Fredholm map between Sobolev spaces

$$(3) \qquad \mathrm{d}_+ + \mathrm{d}^* : L^p_{l+2}(\Lambda^2_+ T^*C) \times L^p_{l+2}(\Lambda^4 T^*C) \longrightarrow L^p_{l+1}(\Lambda^3 T^*C).$$

McLean showed that $\mathrm{d}\tilde{P}$ is onto the image of $\tilde{P}$ and used the Implicit Mapping Theorem for Banach spaces, [4, Thm 1.2.5] to conclude that $\tilde{P}^{-1}(0)$ is smooth, finite-dimensional and locally isomorphic to $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2}\big)$.

Sketch proof of Theorem 3.1. Let (M, φ, g) be an asymptotically cylindrical G_2-manifold asymptotic to $X \times (R, \infty)$, and C an asymptotically cylindrical coassociative 4-fold in M asymptotic to $L \times (R', \infty)$.

Let ν_L be the normal bundle of L in X and $\exp_L : \nu_L \to X$ be the exponential map. For $r > 0$, $B_r(\nu_L)$ is the subbundle of ν_L with fibre at x the open ball about 0 in $\nu_L|_x$ with radius r. Then for small $\epsilon > 0$, there is a tubular neighbourhood T_L of L in X such that $\exp_L : B_{2\epsilon}(\nu_L) \to T_L$ is a diffeomorphism. Also, $\nu_L \times \mathbb{R} \to L \times \mathbb{R}$ is the normal bundle to $L \times \mathbb{R}$ in $X \times \mathbb{R}$ with exponential map $\exp_L \times \mathrm{id} : \nu_L \times \mathbb{R} \to X \times \mathbb{R}$. Then $T_L \times \mathbb{R}$ is a tubular neighborhood of $L \times \mathbb{R}$ in $X \times \mathbb{R}$, and $\exp_L \times \mathrm{id} : B_{2\epsilon}(\nu_L) \times \mathbb{R} \to T_L \times \mathbb{R}$ is a diffeomorphism.

Let $K, R, \Psi : X \times (R, \infty) \to M \setminus K$, and $K', R' > R$, $\Phi : L \times (R', \infty) \to C \setminus K'$, and the normal vector field v on $L \times (R', \infty)$ be as before so that Diagram (1) in Section 2.2 commutes. Then v is a section of $\nu_L \times (R', \infty) \to L \times (R', \infty)$, decaying at rate $O(e^{\beta t})$ and by making K' and R' larger if necessary, we can suppose the graph of v lies in $B_\epsilon(\nu_L) \times (R', \infty)$.

Let $\pi : B_\epsilon(\nu_L) \times (R', \infty) \to L \times (R', \infty)$ be the natural projection. Then we define a diffeomorphism

$$(4) \qquad \Xi : B_\epsilon(\nu_L) \times (R', \infty) \to M \quad \text{by} \quad \Xi : w \mapsto \Psi\big[(\exp_L \times \mathrm{id})(v|_{\pi(w)} + w)\big].$$

where w is a point in $B_\epsilon(\nu_L) \times (R', \infty)$, in the fibre over $\pi(w) \in L \times (R', \infty)$. Under the identification of $L \times (R', \infty)$ with the zero section in $B_\epsilon(\nu_L) \times (R', \infty)$, $\Xi|_{L \times (R', \infty)} \equiv \Phi$. Using Ξ we can then define an isomorphism ξ between the vector bundles $\nu_L \times (R', \infty)$ and $\Phi^*(\nu_C)$ over $L \times (R', \infty)$, where ν_C is the normal bundle of C in M. This leads to the construction of another diffeomorphism $\Theta : B_{\epsilon'}(\nu_C) \to T_C$ for appropriate choices of a small $\epsilon' > 0$, and a tubular neighborhood T_C of C in M.

By choosing the local identification Θ between ν_C and M near C that is compatible with the asymptotic identifications Φ, Ψ of C, M with $L \times \mathbb{R}$ and $X \times \mathbb{R}$ we can then identify submanifolds $\tilde{C}$ of M close to C with small sections of ν_C. More importantly, the asymptotic convergence of $\tilde{C}$ to C, and so to $L \times \mathbb{R}$, is reflected in the asymptotic convergence of sections of ν_C to 0.

We then define a map $Q : L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+ T^*C)\big) \to \{\text{3-forms on } C\}$ by $Q(\zeta^2_+) = (\Theta \circ \zeta^2_+)^*(\varphi)$ for $p > 4$ and $l \geqslant 1$. That is, we regard the section ζ^2_+ as a map $C \to B_{\epsilon'}(\Lambda^2_+ T^*C)$, so $\Theta \circ \zeta^2_+$ is a map $C \to T_C \subset M$, and thus $(\Theta \circ \zeta^2_+)^*(\varphi)$ is a 3-form on C. Therefore if $\Gamma_{\zeta^2_+}$ is the graph of ζ^2_+ in $B_{\epsilon'}(\Lambda^2_+ T^*C)$ and $\tilde{C} = \Theta(\Gamma_{\zeta^2_+})$ its image in M, then $\tilde{C}$ is coassociative if and only if $\varphi|_{\tilde{C}} \equiv 0$, which holds if and only if $Q(\zeta^2_+) = 0$. So basically $Q^{-1}(0)$ parametrizes coassociative 4-folds $\tilde{C}$ close to C. It turns out that $Q : L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+ T^*C)\big) \longrightarrow L^p_{l+1,\gamma}(\Lambda^3 T^*C)$ is a smooth map of Banach manifolds and the linearization of Q at 0 is $\mathrm{d}Q(0) : \zeta^2_+ \mapsto \mathrm{d}\zeta^2_+$.

As in the proof of McLean's Theorem, we then augment Q by a space of 4-forms on C to make it elliptic and define

$$P : L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+ T^*C)\big) \times L^p_{l+2,\gamma}(\Lambda^4 T^*C) \longrightarrow L^p_{l+1,\gamma}(\Lambda^3 T^*C)$$
$$\text{by} \qquad P(\zeta^2_+, \zeta^4) = Q(\zeta^2_+) + \mathrm{d}^* \zeta^4$$

for $p > 4$ and $l \geqslant 1$.

As mentioned before in Section 2.3, on a noncompact manifold, the ellipticity of a differential operator A is not sufficient to ensure that A is Fredholm. Using the analytical framework developed by Lockhart and McOwen in [11] and [12], we showed that in our case $\mathrm{d}P$ is not Fredholm if and only if either $\gamma = 0$, or γ^2 is a positive eigenvalue of $\Delta = \mathrm{d}^*\mathrm{d}$ on functions on L, or γ is an eigenvalue of $- * \mathrm{d}$ on coexact 1-forms on L. Therefore we take γ sufficiently small to guarantee that $\mathrm{d}P$ is Fredholm.

We then show that $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$ is a vector space of smooth, closed, self-dual 2-forms and $\mathrm{Coker}\big((\mathrm{d}^*_+ + \mathrm{d})^q_{l+2,\gamma}\big)$ is a vector space of smooth, closed and coclosed 3-forms. It turns out that the map $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big) \to H^2(C, \mathbb{R})$, $\chi \mapsto [\chi]$ induces an isomorphism of $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$ with a maximal subspace V_+ of the subspace $V \subseteq H^2(C, \mathbb{R})$ on which the cup product $\cup : V \times V \to \mathbb{R}$ is positive definite. Hence

$$(5) \qquad\qquad \dim \mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big) = \dim V_+,$$

which is a topological invariant of C, L.

We finally show that P maps $L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda^2_+ T^*C)\big) \times L^p_{l+2,\gamma}(\Lambda^4 T^*C)$ to the image of $(\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}$ and the image of Q consists of exact 3-forms. The Implicit Mapping Theorem for Banach spaces then implies that $P^{-1}(0)$ is smooth, finite-dimensional and locally isomorphic to $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$. As $P^{-1}(0) = Q^{-1}(0) \times \{0\}$ we conclude that $Q^{-1}(0)$ is also smooth, finite-dimensional and locally isomorphic to $\mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)^p_{l+2,\gamma}\big)$. Moreover, $Q^{-1}(0)$ is independent of l, and so consists of smooth solutions. This proves Theorem 3.1. $\qquad\square$

4. Coassociative Deformations with Moving Boundary and Proof of Theorem 1.1.

We now prove Theorem 1.1. Let (M, φ, g) be an asymptotically cylindrical G_2-manifold asymptotic to $X \times (R, \infty)$, $R > 0$, with decay rate $\alpha < 0$. Let C be an asymptotically cylindrical coassociative 4-fold in X asymptotic to $L \times (R', \infty)$ for $R' > R$ with decay rate β for $\alpha \leqslant \beta < 0$.

As in the proof of Theorem 3.1, we suppose γ satisfies $\beta < \gamma < 0$, and $(0, \gamma^2]$ contains no eigenvalues of the Laplacian Δ_L on functions on L, and $[\gamma, 0)$ contains no eigenvalues of the operator $- * \mathrm{d}$ on coexact 1-forms on L. The reason for this assumption is that, as in the fixed boundary case, by Propositions 4.5, 4.6, we also have the same linearized operator given as

$$(6) \qquad (\mathrm{d}_+ + \mathrm{d}^*)_{l+2,\gamma}^p : F \times (L_{l+2,\gamma}^p(\Lambda_+^2 T^*C) \oplus L_{l+2,\gamma}^p(\Lambda^4 T^*C)) \longrightarrow L_{l+1,\gamma}^p(\Lambda^3 T^*C)$$

for $p > 4$ and $l \geqslant 1$ as in §3.

The conditions on γ imply that $[\gamma, 0) \cap \mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0} = \emptyset$ and hence $\gamma \notin \mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0}$, so that $(\mathrm{d}_+ + \mathrm{d}^*)_{l+2,\gamma}^p$ is Fredholm. Here $\mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0}$ is the discrete set derived from the linearized operator as in [5].

Now, let $K, R, \Psi : X \times (R, \infty) \to M \setminus K$, and $K', R' > R$, $\Phi : L \times (R', \infty) \to C \setminus K'$, and the normal vector field v on $L \times (R', \infty)$ be as before so that Diagram (1) in Section 2 commutes. Let ν_C is the normal bundle of C in M and ν_L be the normal bundle of L in X. Then as in [5], by choosing an appropriate local identification Θ between ν_C and M near C that is compatible with the asymptotic identifications Φ, Ψ of C, M with $L \times \mathbb{R}$ and $X \times \mathbb{R}$ we can identify submanifolds $\tilde{C}$ of M close to C with small sections of ν_C.

Let $F \subset \mathbb{R}^d$ be an open subset of the space of special Lagrangian deformations of L in (X, ω, Ω, g_X). Let $L_0 \in F$ be the starting point and L_s be nearby special Lagrangian submanifolds for some s close to 0 in F. By McLean, [15], the space of deformations $\mathcal{M}_{L_0}$ of a special Lagrangian submanifold L_0 can be identified with closed and coclosed 1-forms on L_0 and so $\mathcal{M}_{L_0}$ corresponds to $H^1(L_0, \mathbb{R})$.

Moreover, one can naturally parametrize L_s by $s \in H^2(L_0, \mathbb{R})$ and relate $\mathcal{M}_{L_0}$ to $H^2(L_0, \mathbb{R})$. Given a compact special Lagrangian L_0, let U be a connected and simply connected open neighbourhood of L_0 in $\mathcal{M}_{L_0}$. One can construct natural local diffeomorphisms $\mathcal{A} : \mathcal{M}_{L_0} \to H^1(L_0, \mathbb{R})$ and $\mathcal{B} : \mathcal{M}_{L_0} \to H^2(L_0, \mathbb{R})$ as follows:

Let $L_s \in U$ be a special Lagrangian submanifold with phase i. Then there exists a smooth path $\tilde{\gamma} : [0, 1] \to U$ with $\tilde{\gamma}(0) = L_0$, and $\tilde{\gamma}(1) = L_s$ which is unique up to isotopy. $\tilde{\gamma}$ parametrizes a family of submanifolds of X diffeomorphic to L_0, and can be lifted to a smooth map $\Gamma : L_0 \times [0, 1] \to X$ with $\Gamma(L_0 \times \{s\}) = \tilde{\gamma}(s)$. Now let $\Gamma^*(\omega)$ be a 2-form on $L_0 \times [0, 1]$. Then $\Gamma^*(\omega)|_{L_0 \times \{s\}} \equiv 0$ for each $s \in [0, 1]$ as each fiber $\tilde{\gamma}(s)$ is Lagrangian. Hence $\Gamma^*(\omega)$ can be written as $\Gamma^*(\omega) = \alpha_s \wedge ds$ where α_s is a closed 1-form on L_0 for $s \in [0, 1]$. Then we integrate α_s with respect to s and take the cohomology class of the closed 1-form $\int_0^1 \alpha_s ds$. So we can define $\mathcal{A}(L_s) = [\int_0^1 \alpha_s ds] \in H^1(L_0, \mathbb{R})$. Similarly, we can write $\Gamma^*(\mathrm{Im}(\Omega)) = \beta_s \wedge ds$ where β_s is a closed 2-form on L_0 for $s \in [0, 1]$ and define $\mathcal{B}(L_s) = [\int_0^1 \beta_s ds] \in H^2(L_0, \mathbb{R})$. For more on the constructions of the $H^2(L_0, \mathbb{R})$ coordinates on the moduli space of special Lagrangian submanifolds see [6] and [8].

We choose a smooth family of diffeomorphisms $\vartheta_s : L_s \to L_0$ such that $\vartheta_0 = \mathrm{id}_{L_0}$ and $L_s \cong L_0$ for small s. Identify a tubular neighbourhood of L_0 in X with a neighbourhood of

the zero-section in T^*L_0, then L_s is the graph of ζ_s, where ζ_s is a small closed and coclosed 2-form on L_0 such that $[\zeta_s] = s$ in $H^2(L_0, \mathbb{R})$.

Having got ζ_s, for some small $s \in F$ we choose sections ϱ_s of $\Lambda_+^2 T^*C$, that depend smoothly on s where $\varrho_0 = 0$ and such that ϱ_s is asymptotic to $\varsigma_s = \zeta_s + dt \wedge *_{L_0}(\zeta_s)$ where $*_{L_0}$ is the star operator in L_0.

One way to do this is to take a smooth function $h : \mathbb{R} \to [0,1]$ defined as $h(x) = 0$ for $x \leq R$ and $h(x) = 1$ for $x \geq R+1$ and set $\varrho_s = \varsigma_s \cdot h(x)$. Then $\varrho_s = \varsigma_s$ in $(R+1, \infty) \times X$ and $\varrho_s = 0$ in K, where $M = K \amalg (R, \infty) \times X$.

Given ζ_s as a section of T^*L_0, we can identify T^*L_0 with $\Lambda_+^2(L_0 \times \mathbb{R})$ and identify this with $\Lambda_+^2(C_0)$ near ∞. Then $\tilde{C}_s$ is the graph of ϱ_s, which is a 4-fold in M asymptotic to $L_s \times \mathbb{R}$ but not necessarily coassociative. So we obtain a family of 4-folds $\tilde{C}_s$ such that $\tilde{C}_0 = C_0 = C$, which is a coassociative submanifold of M, and $\tilde{C}_s$ is asymptotic to $L_s \times \mathbb{R}$, where L_s is special Lagrangian submanifold of X as in Figure 1.
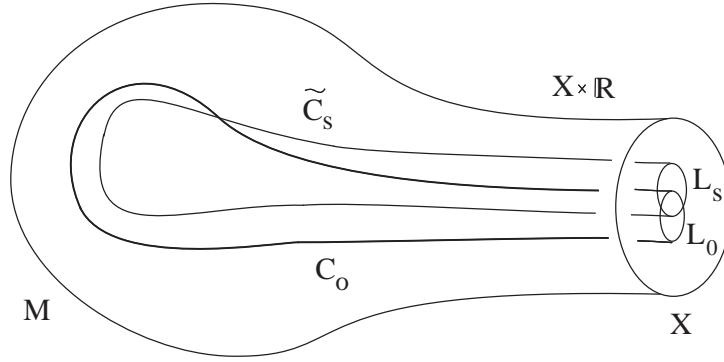

FIGURE 1.

Note that when $\tilde{C}_s$ is asymptotic to $L_s \times (R, \infty)$ near ∞ then $\varphi|_{\tilde{C}_s} \equiv O(e^{\gamma t})$ near ∞ (see Proposition 4.1). On the other hand, since L_s is a special Lagrangian, we can take $\varphi = 0$ in a small neighbourhood of the boundary L_s in $L_s \times (R, \infty)$ and so $\varphi|_{\tilde{C}_s}$ is compactly supported and can take $[\varphi|_{\tilde{C}_s}] \in H^3_{cs}(\tilde{C}_s, \mathbb{R})$. Moreover, since L_0 and L_s are isotopic (which follows from our choice of coordinates in $H^2(L_0, \mathbb{R})$) we can take $[\varphi|_{\tilde{C}_s}] \in H^3_{cs}(C_0, \mathbb{R})$ which is independent of the choice of $\tilde{C}_s$. And $[\varphi|_{\tilde{C}_s}]$ should be zero for there to exist a coassociative 4-fold $\tilde{C}_s$.

Now define $Q : F \times L^p_{l+2,\gamma}\big(B_{\epsilon'}(\Lambda_+^2 T^*C)\big) \to \{\text{3-forms on } C\}$ by $Q(s, \zeta_+^2) = (\Theta \circ (\zeta_+^2 + \varrho_s))^*(\varphi)$. That is, we regard the sum of sections $\zeta_+^2 + \varrho_s$ as a map $C \to B_{\epsilon'}(\Lambda_+^2 T^*C)$, so $\Theta \circ (\zeta_+^2 + \varrho_s)$ is a map $C \to T_C \subset M$, and thus $(\Theta \circ (\zeta_+^2 + \varrho_s))^*(\varphi)$ is a 3-form on C. The point of this definition is that if $\Gamma_{(\zeta_+^2 + \varrho_s))}$ is the graph of $\zeta_+^2 + \varrho_s$ in $B_{\epsilon'}(\Lambda_+^2 T^*C)$ and $\tilde{C}_s = \Theta(\Gamma_{(\zeta_+^2 + \varrho_s)})$ its image in M, then $\tilde{C}_s$ is coassociative if and only if $\varphi|_{\tilde{C}_s} \equiv 0$, which

holds if and only if $Q(s, \zeta_+^2) = 0$. So $Q^{-1}(0)$ parametrizes coassociative 4-folds $\tilde{C}_s$ close to C.

Note that to form $\Gamma(\zeta_+^2 + \varrho_s)$ we have chosen a diffeomorphism $U \subseteq \Lambda_+^2 T^* C_0 = \nu_{C_0} \to V \subseteq M$ where U and V are asymptotic to $U_0 \subset \Lambda_+^2 T^*(L_0 \times \mathbb{R})$ and $V_0 \subset X \times \mathbb{R}$, respectively. Also, we need s to be sufficiently small in F so that $L_s \times \mathbb{R}$ lies in V_0, in other words $L_s \times \mathbb{R}$ is the graph of ς_s in $\Lambda_+^2 T^*(L_0 \times \mathbb{R})$.

Hence we consider $\zeta_+^2 + \varrho_s$ for $\zeta_+^2 \in L_{l+2,\gamma}^p(\Lambda_+^2 T^* C)$ and $\varrho_s \in C^\infty(\Lambda_+^2 T^* C)$ and define

$$Q : F \times L_{l+2,\gamma}^p(\Lambda_+^2 T^* C) \to L_{l+1,\gamma}^p(\Lambda^3 T^* C),$$

$$\text{by} \qquad Q(s, \zeta_+^2) = \pi_*(\varphi|_{\Gamma(\zeta_+^2 + \varrho_s)}).$$

Proposition 4.1. $Q : F \times L_{l+2,\gamma}^p(B_{\epsilon'}(\Lambda_+^2 T^* C)) \longrightarrow L_{l+1,\gamma}^p(\Lambda^3 T^* C)$ *is a smooth map of Banach manifolds. The linearization of Q at 0 is $\mathrm{d}Q(0,0) : (s, \zeta_+^2) \mapsto \mathrm{d}(\zeta_+^2 + \varrho_s)$.*

Proof. The functional form of Q is

$$Q(s, \zeta_+^2)|_x = H(s, x, \zeta_+^2|_x, \nabla \zeta_+^2|_x) \quad \text{for } x \in C,$$

where H is a smooth function of its arguments. Since $p > 4$ and $l \geqslant 1$ by Sobolev embedding theorem we have $L_{l+2,\gamma}^p(\Lambda_+^2 T^* C) \hookrightarrow C_\gamma^1(\Lambda_+^2 T^* C)$. General arguments then show that locally $Q(s, \zeta_+^2)$ is L_{l+1}^p.

From [5], we know that $Q(0, \zeta_+^2)$ lies in $L_{l+1,\gamma}^p(\Lambda^3 T^* C)$. When $s \neq 0$, then $Q(s, 0) = \pi_*(\varphi|_{\Gamma(\varrho_s)})$. By construction $\Gamma(\varrho_s)$ is asymptotic to $L_s \times \mathbb{R}$ which is coassociative in $X \times \mathbb{R}$ as L_s is a special Lagrangian 3-fold. Identify $M \setminus K$ with $X \times \mathbb{R}$. Then φ on $M \setminus K$ can be written as $\varphi = \omega \wedge dt + Re(\Omega) + O(e^\alpha t)$ where α is the rate for M converging to $X \times \mathbb{R}$. $\Gamma(\varrho_s)$ is the graph of ϱ_s which is equal to $L_s \times (R+1, \infty)$ in $X \times (R+1, \infty)$. Since L_s is a special Lagrangian submanifold with phase i, we get for $t > R+1$, $\varphi|_{\Gamma(\varrho_s)} = (\omega \wedge dt + Re(\Omega) + O(e^\alpha t))|_{L_s \times (R+1, \infty)} = 0 + O(e^\alpha t)|_{L_s \times (R+1, \infty)}$.

Therefore for $\Gamma(\varrho_s)$ the error term $\varphi|_{\Gamma(\varrho_s)}$ comes from the degree of the asymptotic decay. In particular, as $\alpha < \gamma$ we can assume $\varphi|_{\Gamma(\varrho_s)} \equiv O(e^{\gamma t})$. We can easily arrange to choose ϱ_s such that $\pi_*(\varphi|_{\Gamma(\varrho_s)}) \in L_{l+1,\gamma}^p(\Lambda^3 T^* C)$ and that $||\pi_*(\varphi|_{\Gamma(\varrho_s)})||_{L_{l+1,\gamma}^p} \leq c|s|$ for some constant c. This implies that $Q(s, \zeta_+^2)$ lies in $L_{l+1,\gamma}^p(\Lambda^3 T^* C)$.

Finally, since we work locally, the linearization of Q is again d as before. $\square$

Next we show that the image of Q consists of exact 3-forms in $L_{l,\gamma}^p(\Lambda^3 T^* C)$.

Proposition 4.2. $Q(F \times L_{l+2,\gamma}^p(B_{\epsilon'}(\Lambda_+^2 T^* C))) \subseteq \mathrm{d}(L_{l+1,\gamma}^p(\Lambda^2 T^* C)) \subset L_{l,\gamma}^p(\Lambda^3 T^* C).$

Proof. This will be an adaptation of a similar proof in [5]. Consider the restriction of the 3-form φ to the tubular neighborhood T_C of C. As φ is closed, and T_C retracts onto C, and $\varphi|_C \equiv 0$, we see that $\varphi|_{T_C}$ is exact. Thus we may write $\varphi|_{T_C} = \mathrm{d}\theta$ for $\theta \in C^\infty(\Lambda^2 T^* T_C)$. Since $\varphi|_C \equiv 0$ we may choose $\theta|_C \equiv 0$. Also, as φ is asymptotic to $O(e^{\beta t})$ with all its derivatives to a translation-invariant 3-form φ_0 on $X \times \mathbb{R}$, we may take θ to be asymptotic to $O(e^{\beta t})$ with all its derivatives to a translation-invariant 2-form on $T_L \times \mathbb{R}$. By Proposition

4.1, the map $(s, \zeta_+^2) \mapsto (\Theta \circ (\zeta_+^2 + \varrho_s))^*(\theta)$ maps $F \times L_{l+2,\gamma}^p \big(B_{\epsilon'}(\Lambda_+^2 T^*C) \big) \to L_{l+1,\gamma}^p(\Lambda^3 T^*C)$. But

$$Q(s, \zeta_+^2) = (\Theta \circ (\zeta_+^2 + \varrho_s))^*(\varphi) = (\Theta \circ (\zeta_+^2 + \varrho_s))^*(\mathrm{d}\theta) = \mathrm{d}\big[(\Theta \circ (\zeta_+^2 + \varrho_s))^*(\theta)\big],$$

so we can conclude that $Q(s, \zeta_+^2) \in \mathrm{d}\big(L_{l+1,\gamma}^p(\Lambda^2 T^*C)\big)$ for $\zeta_+^2 \in L_{l+2,\gamma}^p(B_{\epsilon'}(\Lambda_+^2 T^*C))$ and $\varrho_s \in C^\infty(\Lambda_+^2 T^*C)$. $\qquad\square$

As in [5], we now augment Q by a space of 4-forms on C to make it elliptic. That is, we define
$$P : F \times L_{l+2,\gamma}^p \big(\Lambda_+^2 T^*C \oplus \Lambda^4 T^*C \big) \to L_{l+1,\gamma}^p(\Lambda^3 T^*C),$$
$$\text{by} \qquad P(s, \zeta_+^2, \zeta^4) = \pi_*(\varphi|_{\Gamma(\zeta_+^2 + \varrho_s)}) + \mathrm{d}^* \zeta^4.$$

Note that by the same discussion as in Proposition 4.1, we can also take that $P(s, \zeta_+^2, \zeta^4)$ lies in $L_{l+1,\gamma}^p(\Lambda^3 T^*C)$.

Proposition 4.1 implies that the linearization $\mathrm{d}P$ of P at 0 is the Fredholm operator $(\mathrm{d}_+ + \mathrm{d}^*)_{l+2,\gamma}^p$ of (6). Define $\mathcal{C}$ to be the image of $(\mathrm{d}_+ + \mathrm{d}^*)_{l+2,\gamma}^p$. Then $\mathcal{C}$ is a Banach subspace of $L_{l+1,\gamma}^p(\Lambda^3 T^*C)$, since $(\mathrm{d}_+ + \mathrm{d}^*)_{l+2,\gamma}^p$ is Fredholm. We show P maps into $\mathcal{C}$.

Proposition 4.3. *P maps $F \times L_{l+2,\gamma}^p \big(\Lambda_+^2 T^*C \oplus \Lambda^4 T^*C \big) \longrightarrow \mathcal{C}$.*

Proof. Let $s \in F$ and $(\zeta_+^2, \zeta^4) \in L_{l+2,\gamma}^p \big(B_{\epsilon'}(\Lambda_+^2 T^*C) \big) \times L_{l+2,\gamma}^p(\Lambda^4 T^*C)$, so that $P(s, \zeta_+^2, \zeta^4) = Q(s, \zeta_+^2) + \mathrm{d}^* \zeta^4$ lies in $L_{l+1,\gamma}^p(\Lambda^3 T^*C)$. We must show it lies in $\mathcal{C}$. Since $\gamma \notin \mathcal{D}_{(\mathrm{d}_+ + \mathrm{d}^*)_0}$, this holds if and only if

(7) $$\big\langle Q(s, \zeta_+^2) + \mathrm{d}^* \zeta^4, \chi \big\rangle_{L^2} = 0 \qquad \text{for all } \chi \in \mathrm{Ker}\big((\mathrm{d}_+^* + \mathrm{d})_{m+2,-\gamma}^q\big),$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $m \geqslant 0$.

We proved earlier that $\mathrm{Ker}\big((\mathrm{d}_+^* + \mathrm{d})_{m+2,-\gamma}^q\big)$ consists of closed and coclosed 3-forms χ. We also know that $Q(s, \zeta_+^2) = \mathrm{d}\lambda$ for $\lambda \in L_{l+1,\gamma}^p(\Lambda^2 T^*C)$ by Proposition 4.2. So

$$\big\langle Q(s, \zeta_+^2) + \mathrm{d}^* \zeta^4, \chi \big\rangle_{L^2} = \big\langle \mathrm{d}\lambda, \chi \big\rangle_{L^2} + \big\langle \mathrm{d}^* \zeta^4, \chi \big\rangle_{L^2} = \big\langle \lambda, \mathrm{d}^* \chi \big\rangle_{L^2} + \big\langle \zeta^4, \mathrm{d}\chi \big\rangle_{L^2} = 0$$

for $\chi \in \mathrm{Ker}\big((\mathrm{d}_+^* + \mathrm{d})_{m+2,-\gamma}^q\big)$, as χ is closed and coclosed, and the inner products and integration by parts are valid because of the matching of rates $\gamma, -\gamma$ and L^p, L^q with $\frac{1}{p} + \frac{1}{q} = 1$. So (7) holds, and P maps into $\mathcal{C}$. $\qquad\square$

Proposition 4.3 implies that we can now apply Banach Space Implicit Mapping Theorem, and therefore conclude that $P^{-1}(0)$ is smooth, finite-dimensional and locally isomorphic to $\mathcal{A} = \mathrm{Ker}\big((\mathrm{d}_+ + \mathrm{d}^*)_{l+2,\gamma}^p\big) \subset F \times L_{l+2,\gamma}^p(\Lambda_+^2 T^*C \oplus \Lambda^4 T^*C)$.

Note that our original map was Q and we needed the extra term from the space of 4-forms on C just to make Q elliptic. Therefore, we need to show that the following lemma holds.

Lemma 4.4. *$P^{-1}(0) = Q^{-1}(0) \times \{0\}$.*

Proof. Let's assume that $(s, \zeta_+^2, \zeta^4) \in P^{-1}(0)$, so that $Q(s, \zeta_+^2) + \mathrm{d}^*\zeta^4 = 0$. This should imply $\zeta^4 = 0$, so that $Q(s, \zeta_+^2) = 0$, and therefore $P^{-1}(0) \subseteq Q^{-1}(0) \times \{0\}$. By Proposition 4.2 we have $Q(s, \zeta_+^2) = \mathrm{d}\lambda$ for $\lambda \in L^p_{l+1,\gamma}(\Lambda^2 T^*C)$, so $\mathrm{d}\lambda = -\mathrm{d}^*\zeta^4$. Hence

$$\|\mathrm{d}^*\zeta^4\|^2_{L^2} = \langle \mathrm{d}^*\zeta^4, \mathrm{d}^*\zeta^4 \rangle_{L^2} = -\langle \mathrm{d}^*\zeta^4, \mathrm{d}\lambda \rangle_{L^2} = -\langle \zeta^4, \mathrm{d}^2\lambda \rangle_{L^2} = 0,$$

where the inner products and integration by parts are valid as $L^p_{l+2,\gamma} \hookrightarrow L^2_2$. Thus $Q(s, \zeta_+^2) = \mathrm{d}^*\zeta^4 = 0$. But $\mathrm{d}^*\zeta^4 \cong \nabla\zeta^4$ as ζ^4 is a 4-form, so ζ^4 is constant. Since also $\zeta^4 \to 0$ near infinity in C, we have $\zeta^4 \equiv 0$.

By construction, it is straightforward to show that $Q^{-1}(0) \times \{0\} \subseteq P^{-1}(0)$.

$\square$

Proposition 4.5. *Let $F \subset H^2(L, \mathbb{R})$ be the subspace of special Lagrangian deformations of the boundary L. Also let $\mathrm{d}P_{(0,0,0)}(s, \zeta_+^2, \zeta^4)$ represent the linearization of the deformation map P at 0 with moving boundary and $\mathrm{d}P^f_{(0,0)}(\zeta_+^2, \zeta^4)$ represent the linearization of the deformation map P^f at 0 with fixed boundary. Then*

(8) $$\text{Index of } \mathrm{d}P_{(0,0,0)} = \dim F + \text{index of } \mathrm{d}P^f_{(0,0)}.$$

Proof. At $s = 0$

(9) $$P(0, \zeta_+^2, \zeta^4) = P^f(\zeta_+^2, \zeta^4) = \pi_*(\varphi|_{\Gamma(\zeta_+^2)}) + \mathrm{d}^*\zeta^4$$

Then at $s = 0$, the linearization at $(0, 0, 0)$ is,

(10) $$\mathrm{d}P_{(0,0,0)}(0, \zeta_+^2, \zeta^4) = \mathrm{d}P^f_{(0,0)}(\zeta_+^2, \zeta^4)$$

and since $\mathrm{d}P_{(0,0,0)}$ is linear we have

(11) $$\mathrm{d}P_{(0,0,0)}(s, \zeta_+^2, \zeta^4) = \mathrm{d}P_{(0,0,0)}(s, 0, 0) + \mathrm{d}P_{(0,0,0)}(0, \zeta_+^2, \zeta^4)$$

where $\mathrm{d}P_{(0,0,0)}(s, 0, 0)$ is finite dimensional, $s \in T_0 F = \mathbb{R}^d$.

Then

(12) $$\text{Index of } \mathrm{d}P_{(0,0,0)} : F \times L^p_{l+2,\gamma}\left(\Lambda_+^2 T^*C \oplus \Lambda^4 T^*C\right) \to L^p_{l+1,\gamma}(\Lambda^3 T^*C)$$

(13) $$= \text{index of } \mathrm{d}P^f_{(0,0)} : F \times L^p_{l+2,\gamma}\left(\Lambda_+^2 T^*C \oplus \Lambda^4 T^*C\right) \to L^p_{l+1,\gamma}(\Lambda^3 T^*C)$$

(14) $$= \dim F + (\text{index of } \mathrm{d}P^f_{(0,0)} : L^p_{l+2,\gamma}\left(\Lambda_+^2 T^*C \oplus \Lambda^4 T^*C\right) \to L^p_{l+1,\gamma}(\Lambda^3 T^*C)).$$

$\square$

In Proposition 4.6 we determine the set F. First, we construct a linear map $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{cs}(C)$ and then show that F should be restricted to the kernel of Υ.

Proposition 4.6. *There is a linear map* $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{\mathrm{cs}}(C)$ *given explicitly as* $\Upsilon(\varsigma) :$ $[\varsigma] \rightarrowtail [\mathrm{d}\tilde{\varsigma}]$ *and* F *is an open subset of* $\ker(\Upsilon)$.

Proof. F is a subset of the special Lagrangian 3-fold deformations of L. So we could take $F \subset H^2(L, \mathbb{R})$. We can construct a linear map $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{\mathrm{cs}}(C)$ which comes from the standard long exact sequence in cohomology:

$$(15) \quad \begin{array}{c} 0 \longrightarrow H^0(C) \longrightarrow H^0(L) \longrightarrow H^1_{\mathrm{cs}}(C) \longrightarrow H^1(C) \longrightarrow H^1(L) \longrightarrow H^2_{\mathrm{cs}}(C) \\ \Big\downarrow \\ 0 \longleftarrow H^4_{\mathrm{cs}}(C) \longleftarrow H^3(L) \longleftarrow H^3(C) \longleftarrow H^3_{\mathrm{cs}}(C) \longleftarrow H^2(L) \longleftarrow H^2(C) \end{array}$$

where $H^k(C) = H^k(C, \mathbb{R})$ and $H^k(L) = H^k(L, \mathbb{R})$ are the de Rham cohomology groups, and $H^k_{\mathrm{cs}}(C, \mathbb{R})$ is the compactly-supported de Rham cohomology group.

The explicit definition of the map Υ is that if ς is a closed 2-form on L then extend ς smoothly to a 2-form $\tilde{\varsigma}$ on C which is asymptotic to ς on $L \times (\mathbb{R}, \infty)$. Then $\mathrm{d}\tilde{\varsigma}$ is a closed compactly supported 3-form on C. So $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{\mathrm{cs}}(C)$ can be defined explicitly as $\Upsilon(\varsigma) : [\varsigma] \rightarrowtail [\mathrm{d}\tilde{\varsigma}]$. Also note that by construction of the coordinates in $H^2(L, \mathbb{R})$, $[\varphi|_{\tilde{C}_s}] = \Upsilon([s])$, for $s \in H^2(L, \mathbb{R})$ and for the graphs of ϱ_s.

The potential problem here is that the image of Υ may not be zero in $H^3_{\mathrm{cs}}(C)$. In [5], we studied the properties of the kernel and cokernel of the operator $d_+ + d^*$. The dimension of the cokernel is

$$\dim \mathrm{Ker}\big((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2, -\gamma}\big) = b^0(L) - b^0(C) + b^1(C).$$

The map $\mathrm{Ker}\big((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2, -\gamma}\big) \to H^3(C, \mathbb{R})$, $\chi \mapsto [\chi]$ is surjective, with kernel of dimension $b^0(L) - b^0(C) + b^1(C) - b^3(C) \geqslant 0$, which is the dimension of the kernel of $H^3_{\mathrm{cs}}(C, \mathbb{R}) \to H^3(C, \mathbb{R})$. So we can think of $\mathrm{Ker}\big((\mathrm{d}^*_+ + \mathrm{d})^q_{m+2, -\gamma}\big)$ as a space of closed and coclosed 3-forms filling out all of $H^3_{\mathrm{cs}}(C, \mathbb{R})$ and $H^3(C, \mathbb{R})$. In other words cokernel has a piece looking like the kernel of $H^3_{\mathrm{cs}}(C, \mathbb{R}) \to H^3(C, \mathbb{R})$. This is equivalent to the image of $\Upsilon : H^2(L, \mathbb{R}) \to H^3_{\mathrm{cs}}(C)$ by exactness of the long exact sequence (15).

So the problem here is that if we allow the boundary to move arbitrarily then the image of Υ may not be zero in $H^3_{\mathrm{cs}}(C)$. This means that, for $s \in H^2(L, \mathbb{R})$, the construction of asymptotically cylindrical coassociative 4-fold asymptotic to $L_s \times (R, \infty)$ is obstructed if and only if $\Upsilon(s) \neq 0$. Therefore the set F is restricted to $\ker \Upsilon \subset H^2(L, \mathbb{R})$.

$\square$

Next, we determine the dimension of the moduli space $\mathcal{M}^\gamma_C$.

Proposition 4.7. *The dimension of* $\mathcal{M}^\gamma_C$, *the moduli space of coassociative deformations of an asymptotically cylindrical coassociative submanifold* C *asymptotic to* $L_f \times (R, \infty)$, $f \in F$, *with decay rate* γ *is*

$$(16) \qquad \dim(\mathcal{M}^\gamma_C) = \dim(V_+) + b^2(L) - b^0(L) + b^0(C) - b^1(C) + b^3(C).$$

Proof. Let $b^k(C)$, $b^k(L)$ and $b^k_{cs}(C)$ be the corresponding Betti numbers as before. The actual dimension of the moduli space for free boundary is the sum of the dimension of the fixed boundary and the dimension of the kernel of Υ.

We know from Propositions 4.5, and 4.6 that

(17) $$\text{index of } dP_{(0,0,0)} = \dim F + \text{index of } dP^f_{(0,0)}.$$

In particular,

(18) $$\dim \left(\ker dP_{(0,0,0)} \right) = \dim \left(\ker \Upsilon \right) + \dim \left(V_+ \right).$$

Taking alternating sums of dimensions in (15), shows that the dimension of the kernel of Υ is

(19)
$$\begin{aligned}
\dim \left(\ker \Upsilon \right) &= b^2(L) - \text{Im}(\Upsilon) \\
&= b^2(L) - [b^0(L) - b^0(C) + b^1(C) - b^3(C)].
\end{aligned}$$

Therefore the dimension of $\mathcal{M}^\gamma_C$, the moduli space of coassociative deformations of an asymptotically cylindrical coassociative submanifold C asymptotic to $L_s \times (R, \infty)$, $s \in F$, with decay rate γ is

(20)
$$\begin{aligned}
\dim(\mathcal{M}^\gamma_C) &= \dim(V_+) + \dim(\ker \Upsilon). \\
&= \dim(V_+) + b^2(L) - b^0(L) + b^0(C) - b^1(C) + b^3(C).
\end{aligned}$$

$\square$

And finally straightforward modifications of Propositions 4.6-4.8 in [5] provides us the standard elliptic regularity results and the bootstrapping argument to conclude that $Q^{-1}(0)$ is a smooth, finite dimensional manifold, and so we complete the proof of Theorem 1.1. Here we will skip these details to avoid repetition, for more on regularity results see [5].

5. Applications: Topological Quantum Field Theory of Coassociative Cycles

In [10], it was mentioned that if one could show the analytical details of the deformation theory of asymptotically cylindrical coassociative submanifolds inside a G_2-manifold then it would be possible to study global properties of these moduli spaces. In particular, one needs a result like Theorem 1.1 which shows that $H^2_+(C, L)$ parametrizes the coassociative deformations of C with boundary L. In this section using Theorem 1.1 and a well-known theory of (anti) self-dual connections (for more on the subject see [2], [3]) we will verify this claim.

Here are some basic definitions:

Let (M, φ, g) be an asymptotically cylindrical G_2-manifold asymptotic to $X \times (R, \infty)$, and C an asymptotically cylindrical coassociative 4-fold in M asymptotic to $L \times (R', \infty)$. Let E be a fixed rank one vector bundle over C. A connection D_E on C has finite energy if

$$\int_C |F_E|^2 \, dV < \infty$$

where F_E is the curvature of the connection D_E and dV is the volume form with respect to the metric. Note that connections with finite energy play an important role in Yang-Mills Theory.

Definition 5.1. Let $(X, \omega, J, g_X, \Omega)$ be a Calabi–Yau 3-fold and L be a special Lagrangian submanifold of X. Also let (M, φ, g_M) be a G_2-manifold asymptotic to $X \times (R, \infty)$. A pair (C, D_E) is called a coassociative cycle if C is a coassociative submanifold of M asymptotic to $L \times (R', \infty)$ and D_E is an anti-self-dual, unitary connection over C with finite energy.

Definition 5.2. Let $(X, \omega, J, g_X, \Omega)$ be a Calabi–Yau 3-fold and L be a special Lagrangian submanifold in X. A pair $(L, D_{E'})$ is called a special Lagrangian cycle if $D_{E'}$ is a unitary flat connection over L induced from D_E.

Let $\mathcal{M}^{slag}(X)$ denote the moduli space of special Lagrangian cycles in X. In [8], Hitchin proved the following theorem:

Theorem 5.3. *The tangent space to $\mathcal{M}^{slag}(X)$ is naturally identified with the space $H^2(L, \mathbb{R}) \times H^1(L, ad(E'))$. For line bundles over L, the cup product $\cup : H^2(L, \mathbb{R}) \times H^1(L, \mathbb{R}) \longrightarrow \mathbb{R}$ induces a symplectic structure on $\mathcal{M}^{slag}(X)$.*

Now using Theorem 1.1 we verify the proof of the following theorem which was claimed by Leung [Claim 10, Sec.4] in [10].

Theorem 5.4. *Let X be a Calabi–Yau 3-fold and L be a special Lagrangian submanifold in X. Let M be a G_2-manifold asymptotic to $X \times (R, \infty)$, and C a coassociative 4-fold in M asymptotic to $L \times (R', \infty)$. Let also $\mathcal{M}^{coas}(M)$ be the moduli space of coassociative cycles in M and $\mathcal{M}^{slag}(X)$ be the moduli space of special Lagrangian cycles in X. Then the boundary map*

(21)
$$b : \mathcal{M}^{coas}(M) \longrightarrow \mathcal{M}^{slag}(X)$$
$$b(C, D_E) = (L, D_{E'})$$

is a Lagrangian immersion.

Proof. Let $\delta : H^1(L) \to H^2_+(C, L)$, $j^* : H^2_+(C, L) \longrightarrow H^2_+(C)$, $i^* : H^2_+(C) \longrightarrow H^2(L)$ be the canonical maps. They give dual maps $i_* : H_2(L) \longrightarrow H^+_2(C)$, $j_* : H^+_2(C) \longrightarrow H^+_2(C, L)$ and $\partial : H^+_2(C, L) \longrightarrow H_1(L)$. Then we have the following long exact sequences:

$$
\begin{array}{ccccccccccc}
\ldots \longrightarrow H^1(L) & \longrightarrow & H^2_+(C, L) & \longrightarrow & H^2_+(C) & \longrightarrow & H^2(L) & \longrightarrow & H^3(C, L) & \longrightarrow \ldots \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
\ldots \longrightarrow H_2(L) & \longrightarrow & H^+_2(C) & \longrightarrow & H^+_2(C, L) & \longrightarrow & H_1(L) & \longrightarrow & H_1(C) & \longrightarrow \ldots
\end{array}
$$

Note that the classes coming from the boundary L should have zero self-intersection. So we can rewrite the long exact sequences above starting from 0, instead of $H^1(L)$ and $H_2(L)$.

From the sequences we get

$$\ker \delta \cong \ker i_* \cong im\partial \cong H_2^+(C,L)/\ker \partial = H_2^+(C,L)/im(j_*)$$

which implies

(22)
$$\dim (\ker \delta) = \dim (H_2^+(C,L)) - \dim(im(j_*))$$

Note that the only classes that survive in $H_2^+(C) \longrightarrow H_2^+(C,L)$, (i.e. do not go to zero) have self intersection 0. So one can identify $im(j_*)$ with $H_2^+(C)$.

This implies that there is a short exact sequence

$$0 \longrightarrow im(j_*) \longrightarrow H_2^+(C,L) \longrightarrow im\partial \longrightarrow 0,$$

or equivalently

$$0 \longrightarrow H_2^+(C) \longrightarrow H_2^+(C,L) \longrightarrow \ker \delta \longrightarrow 0.$$

Then as a consequence of (22) and Theorem 1.1 we conclude that $H_+^2(C,L) = V_+ \oplus \ker \delta$ parametrizes the deformations of C with moving boundary ∂C. From our previous work in [5], it follows that $H_+^2(C) \cong V_+$ parametrizes the deformations of C with fixed boundary and McLean showed that $H^2(L)$ gives the special Lagrangian deformations of L.

Note that the linearization of the boundary value map $\mathcal{M}^{coas}(M) \longrightarrow \mathcal{M}^{slag}(X)$ in Theorem 5.4 is given by $i^* : H_+^2(C) \longrightarrow H^2(L)$ and for the connection part $\beta : H^1(C) \longrightarrow H^1(L)$. It is straightforward that $\mathrm{Im}(i^*) \oplus \mathrm{Im}\beta$ is a subspace of $H^2(L) \oplus H^1(L)$. By definition of cup product, the symplectic structure reduces to 0 on $\mathrm{Im}(i^*) \oplus \mathrm{Im}\beta$ and by Poincaré Duality, $\dim(\mathrm{Im}(i^*) \oplus \mathrm{Im}\beta) = \frac{1}{2} \dim(H^2(L) \oplus H^1(L))$.

Thus we conclude that $\mathrm{Im}(i^*) \oplus \mathrm{Im}\beta$ is a Lagrangian subpace of $H^2(L) \oplus H^1(L)$ with the symplectic structure defined above and conclude the proof of Theorem 5.4.

$\square$

REFERENCES

[1] Akbulut, S. and Salur, S., *Deformations in G_2-Manifolds* math.GT/0701790.
[2] Donaldson, S.K. and Kronheimer, P.B., *The geometry of four-manifolds.* Oxford Math. Monographs, OUP, New York, 1990.
[3] Freed, D.S. and Uhlenbeck, K.K., *Instantons and Four-Manifolds*, MSRI Publications, Volume 1, Springer-Verlag 1984.
[4] Joyce, D.D., *Compact Manifolds with Special Holonomy*, OUP, Oxford, 2000.
[5] Joyce, D.D. and Salur, S., *Deformations of Asymptotically Cylindrical Coassociative Submanifolds with Fixed Boundary*, Geometry&Topology, Vol. 9 (2005) Paper no. 25, 1115–1146.
[6] Gross, M., Huybrechts, D. and Joyce, D.D., *Calabi-Yau manifolds and Related Geometries*, Springer, 2003.
[7] Harvey, F.R. and Lawson, H.B., *Calibrated Geometries*, Acta. Math. **148** (1982), 47–157.
[8] Hitchin, N. *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), no. **3-4**, 503–515 (1998).

[9] Kovalev, A.G., *Twisted connected sums and special Riemannian holonomy*, J. reine angew. Math. **565** (2003), 125–160. math.DG/0012189.

[10] Leung, C.L., *Topological Quantum Field Theory for Calabi–Yau threefolds and G_2-manifolds*, Adv. Theor. Math. Phys. **6** (2002), no. 3, 575–591.

[11] Lockhart, R.B., *Fredholm, Hodge and Liouville Theorems on noncompact manifolds*, Trans. A.M.S. **301** (1987), 1–35.

[12] Lockhart, R.B. and McOwen, R.C., *Elliptic differential operators on noncompact manifolds*, Ann. Scuola norm. sup. Pisa, Classe di scienze **12** (1987), 409–447.

[13] Melrose, R., *The Atiyah-Patodi-Singer Index Theorem*, A.K. Peters, Wellesley, MA, 1994.

[14] Melrose, R., *Spectral and Scattering Theory for the Laplacian on Asymptotically Euclidean Spaces*, in Ikawa M., editor, *Spectral and Scattering Theory*, Lecture notes in Pure and Applied Mathematics, vol. **161**, Marcel Dekker Inc., 1994.

[15] McLean, R.C., *Deformations of calibrated submanifolds*, Comm. Anal. Geom. **6** (1998), 705–747.

[16] Salur, S., *Asymptotically Cylindrical Ricci-Flat Manifolds*, Proc. Amer. Math. Soc., 134 (2006), no. 10, 3049–3056.

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